Name: Solutions

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Multiple Choice

1.(6 pts.) The function $f(x) = 2x + \ln x$ is one-to-one. Compute $(f^{-1})'(2)$.

~~(a)~~ $\frac{1}{3}$ (b) $\frac{5}{2}$ (c) $4 + \ln 2$

(d) $\frac{2}{5}$ (e) 0

First, $f'(x) = 2 + \frac{1}{x} > 0$ for $x > 0$.

$f(x)$ is increasing and hence 1-1 for $x > 0$.

Next, note $f(1) = 2$. Hence

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{2 + \frac{1}{1}} = \frac{1}{3}$$

2.(6 pts.) Solve the equation $\log_4(x) + \log_4(x^2) = -\frac{3}{2}$. Then $x =$

(a) $\frac{1}{\sqrt{e}}$ (b) 2 ~~(c)~~ $\frac{1}{2}$ (d) $\frac{3}{2}$ (e) (e) -2

First $\log_4 x^3 = -\frac{3}{2}$

$\Rightarrow x^3 = 4^{-\frac{3}{2}}$

$\Rightarrow x = 4^{-\frac{1}{2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$

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3.(6 pts.) Use logarithmic differentiation to compute the derivative of the function

$$f(x) = \frac{2^x(x^3+1)}{\sqrt{x+1}}.$$

~~(a)~~ $f'(x) = \frac{2^x(x^3+1)}{\sqrt{x+1}}\left(\ln 2 + \frac{3x^2}{x^3+1} - \frac{1}{2(x+1)}\right)$

(b) $f'(x) = \frac{2^x(x^3+1)}{\sqrt{x+1}}\left(\frac{1}{2} + \frac{3x^2}{x^3+1} - \frac{1}{2(x+1)}\right)$

(c) $f'(x) = \frac{2^x(x^3+1)}{\sqrt{x+1}}\left(\frac{1}{\ln 2} + \frac{1}{x^3+1} - \frac{1}{x+1}\right)$

(d) $f'(x) = \frac{2^x(x^3+1)}{\sqrt{x+1}}\left(2 + \frac{1}{x^3+1} - \frac{1}{x+1}\right)$

(e) $f'(x) = \frac{2^x(x^3+1)}{\sqrt{x+1}}\left(\frac{1}{\ln 2} + \frac{3x^2}{x^3+1} - \frac{1}{2(x+1)}\right)$

$$\ln f(x) = x\ln 2 + \ln(x^3+1) - \frac{1}{2}\ln(x+1)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = \ln 2 + \frac{3x^2}{x^3+1} - \frac{1}{2(x+1)}$$

4.(6 pts.) You begin an experiment at 9am with a sample of 1000 bacteria. An hour later your population has doubled. Assuming exponential growth, what is the population at noon?

(a) 32,000 (b) 4,000 ~~(c)~~ 8,000 (d) $1{,}000e^{-3}$ (e) $1{,}000e^3$

Note: $y(t) = 1000e^{rt}$

$y(1) = 1000e^{r} = 2000 \Rightarrow e^r = 2.$

At noon: $y(3) = 1000e^{3r} = 1000(e^r)^3 = 1000 \cdot 2^3$

$= 8000$

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5.(6 pts.) Simplify $\sin^{-1}(\sin \frac{9\pi}{10})$.

(a) $-\frac{\pi}{10}$

(b) $\frac{9\pi}{10}$

(c) not enough information to tell.

~~(d)~~ $\frac{\pi}{10}$

(e) 0

Note $\sin \frac{9\pi}{10} = \sin \frac{\pi}{10}$ and $\frac{\pi}{10} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Then $\sin^{-1}(\sin \frac{9\pi}{10}) = \sin^{-1}(\sin \frac{\pi}{10}) = \frac{\pi}{10}$.

6.(6 pts.) Compute the limit $\lim_{x\to\infty} (2x)^{\frac{1}{x}}$.

~~(a)~~ 1 (b) 0 (c) e (d) 2 (e) ∞

$$(2x)^{\frac{1}{x}} = e^{\ln (2x)^{\frac{1}{x}}} = e^{\frac{\ln(2x)}{x}}$$

$$\lim_{x\to\infty} \frac{\ln(2x)}{x} = \lim_{x\to\infty} \frac{\frac{2}{2x}}{1} = 0.$$

$$\Rightarrow \lim_{x\to\infty} (2x)^{\frac{1}{x}} = e^0 = 1.$$

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7.(6 pts.) Evaluate the integral

$$\int_0^{\frac{\pi}{2}} x\sin(x)dx.$$

(a) 0 (b) 1 (c) -1 (d) π (e) $\frac{\pi}{2}$

Integration by parts:

$$\int x \sin x\, dx = -\int x\, d\cos x = -\left[x\cos x - \int \cos x\, dx\right]$$

$$= -\left[x\cos x - \sin x + C\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} x\sin x\, dx = -\left(x\cos x - \sin x\right)\Big|_0^{\frac{\pi}{2}}$$

$$= -\left(\frac{\pi}{2}\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right) + \left(0\cos 0 - \sin 0\right) = 1$$

8.(6 pts.) Find the integral $\int_0^2 \sqrt{4-x^2}dx$.

(a) 0 (b) $4+\sin 4$ (c) π (d) $\pi+2$ (e) -2

$x = 2\sin\theta \Rightarrow \sqrt{4-x^2} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

$0 \leq x \leq 2 \Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$

$$\int_0^2 \sqrt{4-x^2}\, dx = \int_0^{\frac{\pi}{2}} 2\cos\theta \cdot 2\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} 4\cos^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2(1+\cos 2\theta)\, d\theta = (2\theta + \sin 2\theta)\Big|_0^{\frac{\pi}{2}} = \pi.$$

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9.(6 pts.) Evaluate the integral

$$\int \frac{x-9}{x^2+3x-10}\,dx.$$

(a) $\ln\left|\frac{(x-5)^2}{x+2}\right|+C$ (b) $\ln\left|\frac{x-2}{x+5}\right|+C$ (c) $\ln\left|\frac{(x+2)^2}{(x-5)^3}\right|+C$

(d) $\ln\left|\frac{(x+5)^2}{x-2}\right|+C$ (e) $\ln\left|\frac{x+5}{(x-2)^2}\right|+C$

$x^2+3x-10=(x+5)(x-2)$

Set $\frac{x-9}{x^2+3x-10}=\frac{A}{x+5}+\frac{B}{x-2} \Rightarrow x-9=A(x-2)+B(x+5)$

$x=2 \Rightarrow B=-1$; $x=-5 \Rightarrow A=2$

$\Rightarrow \frac{x-9}{x^2+3x-10}=\frac{2}{x+5}-\frac{1}{x-2} \Rightarrow \int\frac{x-9}{x^2+3x-10}dx=\ln(x+5)^2-\ln|x-2|+C$

10.(6 pts.) Determine whether the following integral converges or diverges. If it converges, evaluate.

$$\int_{-2}^{0} \frac{1}{(x+1)^2}\,dx.$$

(a) Converges to -2. (b) Converges to 0. (c) Converges to 2.

(d) Diverges. (e) Converges to 1.

Check $\int_{-1}^{0}\frac{1}{(x+1)^2}dx=\lim_{t\to-1^+}\int_t^0\frac{1}{(x+1)^2}dx$

$=\lim_{t\to-1^+}\left(-\frac{1}{x+1}\right)\Big|_t^0=\lim_{t\to-1^+}\left(\frac{1}{t+1}-1\right)=\infty$

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11.(6 pts.) Find the length of the curve

$$y = 1 + \frac{4}{3}x^{3/2} \qquad \text{for } 0 \le x \le 1.$$

(a) $\frac{1}{6}(1-\sqrt{5})$ (b) $\frac{2}{3}(1-\sqrt{5})$ (c) $\frac{1}{12}(3\sqrt{3}-1)$

(d) $\frac{1}{12}(11\sqrt{5}-1)$ ~~(e)~~ $\frac{1}{6}(5\sqrt{5}-1)$

$1+(y')^2 = 1+\left(\frac{4}{3}\cdot\frac{3}{2}x^{\frac{1}{2}}\right)^2 = 1+4x$.

arc length $= \int_0^1 \sqrt{1+(y')^2}\, dx = \int_0^1 \sqrt{1+4x}\, dx = \frac{2}{3}\cdot\frac{1}{4}(1+4x)^{\frac{3}{2}}\Big|_0^1$

$= \frac{1}{6}\left[5^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{1}{6}\left[5\sqrt{5}-1\right]$

12.(6 pts.) Find the centroid of the region bounded by $y = x^2$ and $y = x$.

(a) $(\frac{1}{2}, \frac{1}{2})$ (b) $(\frac{1}{10}, \frac{2}{5})$ (c) $(\frac{1}{10}, \frac{1}{15})$

(d) $(\frac{1}{12}, \frac{1}{15})$ ~~(e)~~ $(\frac{1}{2}, \frac{2}{5})$

Intersections: $x^2 = x \Rightarrow x=0, \quad x=1$.

$(0,0) \quad (1,1)$.

$A = \int_0^1 (x-x^2)\,dx = \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$.

$\bar{x} = 6\int_0^1 x(x-x^2)\,dx = 6\int_0^1 (x^2-x^3)\,dx = 6\left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1 = 6\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{1}{2}$.

$\bar{y} = 6\int_0^1 \frac{1}{2}\left[x^2 - x^4\right]dx = 3\int_0^1 (x^2-x^4)\,dx = 3\left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1$

$= 3\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{2}{5}$

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13.(6 pts.) The solution to the initial value problem

$$y' = \frac{\sin(x)}{2y+1} \qquad y(0) = 2$$

satisfies

(a) $2y + 1 = 6 - e^{-\cos x}$ (b) $y^2 + y = 7 - \cos x$

(c) $y^2 + y = 6\cos x$ (d) $2y + 1 = 5e^{-\cos x}$

(e) $e^{2y+1} = e^5 + \arcsin x$

Separable equation: $(2y+1)dy = \sin x \, dx$

$y^2 + y = -\cos x + C$

$x=0,\ y=2 \Rightarrow 4+2 = -1 + C \Rightarrow C = 7$

$\Rightarrow y^2 + y = 7 - \cos x$

14.(6 pts.) The solution to the initial value problem

$$\frac{dy}{dx} + xy + x = 0 \qquad y(0) = 0$$

is

(a) $y = e^{-\frac{x^2}{2}} - 1$ (b) $y = e - e^{-\frac{x^2}{2}+1}$ (c) $y = xe^x$

(d) $y = 1 - e^{-x}$ (e) $y = e^{-x} - 1$

$\frac{dy}{dx} + xy = -x$

$I(x) = e^{\int x\,dx} = e^{\frac{1}{2}x^2} \Rightarrow (e^{\frac{1}{2}x^2}y)' = -xe^{\frac{1}{2}x^2}$

$\Rightarrow e^{\frac{1}{2}x^2}y = -\int xe^{\frac{1}{2}x^2}dx = -e^{\frac{1}{2}x^2} + C$

$\Rightarrow y = Ce^{-\frac{1}{2}x^2} - 1$

$x=0,\ y=0 \Rightarrow 0 = C - 1 \Rightarrow C = 1 \Rightarrow y = e^{-\frac{1}{2}x^2} - 1$

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15.(6 pts.) Investigate the convergence or divergence of the sequence

$$\lim_{n\to\infty}(-1)^n\frac{3n^2}{n^2+1}.$$

If the sequence converges, find its limit.

(a) -3 (b) $(-1)^n 3$ ~~(c)~~ The sequence is divergent

(d) 3 (e) 0

$$\lim_{n\to\infty}\frac{3n^2}{n^2+1}=3 \quad \Rightarrow \quad \lim_{n\to\infty}(-1)^n\frac{3n^2}{n^2+1}\neq 0$$

Use Divergence Test

16.(6 pts.) Investigate convergence or divergence of the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(4-\pi)^{n-1}}{\pi^n}.$$

If the series converges, calculate its sum. Note: $4>\pi>3$.

(a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{4}$ (c) The series is divergent

~~(d)~~ $\frac{1}{4}$ (e) $-\frac{1}{4}$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(4-\pi)^{n-1}}{\pi^n}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(4-\pi)^{n-1}}{\pi\cdot\pi^{n-1}}$$

$$=\frac{1}{\pi}\sum_{n=1}^{\infty}\left(-\frac{4-\pi}{\pi}\right)^{n-1}$$ geometric series

$$=\frac{1}{\pi}\cdot\frac{1}{1-\left(-\frac{4-\pi}{\pi}\right)}=\frac{1}{\pi}\cdot\frac{1}{1+\frac{4-\pi}{\pi}}=\frac{1}{4}$$

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17.(6 pts.) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \cos\left(\frac{1}{n}\right)$$

is

~~(a)~~ absolutely convergent by limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

(b) conditionally convergent by root test

(c) divergent by integral test

(d) divergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

(e) absolutely convergent by ratio test

Note $\left|\frac{1}{n^{\frac{3}{2}}} \cos\left(\frac{1}{n}\right)\right| \leq \frac{1}{n^{\frac{3}{2}}}$ Use comparison test

18.(6 pts.) Which of the following series converge conditionally?

$$(1) \sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n}+1}; \qquad (2) \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}+1}; \qquad (3) \sum_{n=0}^{\infty} (-1)^n \frac{1}{n^{5/2}+1}.$$

(a) (1) and (2) converge conditionally, (3) does not converge conditionally

(b) (2) converges conditionally, (1) and (3) do not converge conditionally

~~(c)~~ (1) converges conditionally, (2) and (3) do not converge conditionally

(d) (1) and (3) converge conditionally, (2) does not converge conditionally

(e) (3) converges conditionally, (1) and (2) do not converge conditionally

(2) is a positive series and CANNOT be cond. conv.

(3): positive part: $\sum \frac{1}{n^{\frac{5}{2}}+1} \sim \sum \frac{1}{n^{\frac{5}{2}}}$

$\Rightarrow$ (3) converges absolutely

(1): Alternating series test to get convergence.

NOT absolutely convergent by comparison with $\sum \frac{1}{\sqrt{n}}$.

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19.(6 pts.) The series

$$\sum_{n=1}^{\infty} \frac{8^n}{n^2}(x-1)^{3n}$$

has the radius of convergence

(a) 0 ~~(b)~~ $\frac{1}{2}$ (c) ∞ (d) 1 (e) $\frac{1}{8}$

$$\lim_{n\to\infty}\left|\frac{\frac{8^{n+1}}{(n+1)^2}(x-1)^{3(n+1)}}{\frac{8^n}{n^2}(x-1)^{3n}}\right| = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\,8(x-1)^3\right| = 8|x-1|^3 < 1$$

$$\Leftrightarrow |x-1|^3 < \frac{1}{8} \Leftrightarrow |x-1| < \frac{1}{2}$$

20.(6 pts.) Consider the Taylor series of

$$f(x) = \sum_{n=1}^{\infty} \frac{n^n}{n!}x^n.$$

Find $f^{(100)}(0)$.

(a) $\frac{100^{100}}{((100)!)^2}$ (b) $\frac{(100)!}{100^{100}}$ (c) $\frac{100^{100}}{(100)!}$

(d) $(100)!$ ~~(e)~~ 100^{100}

$$\frac{100^{100}}{100!} = \frac{1}{100!}f^{(100)}(0)$$

$$\Rightarrow f^{(100)}(0) = 100^{100}$$

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21.(6 pts.) Which is the only statement that is true about the three series

$$\text{(I)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+n^2} \quad \text{(II)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{1+n^2} \quad \text{(III)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ln n}{1+n^2} \ ?$$

(a) (I) and (III) converge conditionally, (II) converges absolutely

(b) (I) diverges, (II) converges conditionally, (III) converge absolutely

(c) (I) and (II) converge absolutely, (III) converges conditionally

(d) (I) and (III) converge conditionally, (II) diverges

~~(e)~~ (I) and (III) converge absolutely, (II) converges conditionally

(I) converges absolutely by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(II): converges itself by alternating series test
not absolutely convergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$

(III) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges by integral test

$\Rightarrow$ (III) converges absolutely

22.(6 pts.) Let $x = \sin(9t)$ and $y = \cos(9t)$. Then $\frac{dy}{dx} =$

(a) $\tan(9t)$ ~~(b)~~ $-\tan(9t)$ (c) $9\tan(t)$

(d) $81\sec^2(9t)$ (e) $\cot(9t)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-9\sin(9t)}{9\cos(9t)} = -\tan(9t)$$

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23.(6 pts.) The point $(2, \frac{13\pi}{6})$ in polar coordinates corresponds to which point below in Cartesian coordinates?

~~(a)~~ $(\sqrt{3}, 1)$

(b) $(-\sqrt{3}, 1)$

(c) $(1, \sqrt{3})$

(d) $(-1, \sqrt{3})$

(e) Since $\frac{13\pi}{6} > 2\pi$, there is no such point.

$$x = 2\cos\frac{13\pi}{6} = 2\cos\frac{\pi}{6} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = 2\sin\frac{13\pi}{6} = 2\sin\frac{\pi}{6} = 2\cdot\frac{1}{2} = 1$$

24.(6 pts.) Which integral below gives the surface area of the surface of revolution obtained by rotating the polar curve $r = \sin\theta$, $0 \le \theta \le \pi$ about the x-axis?
Hint: A polar curve is also a parameterized curve.

(a) $2\pi \int_0^{\pi} \sin\theta\ \cos^2\theta\, d\theta$

(b) $2\pi \int_0^{\pi} \cos^2\theta\, d\theta$

~~(c)~~ $2\pi \int_0^{\pi} \sin^2\theta\, d\theta$

(d) $\frac{\pi}{2} \int_0^{\pi} \sin\theta\ \cos^2\theta\, d\theta$

(e) $2\pi \int_0^{\pi} \sin\theta\ \cos\theta\, d\theta$

$$x = r\cos\theta = \sin\theta\cos\theta \Rightarrow x' = \cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$y = r\sin\theta = \sin^2\theta \Rightarrow y' = 2\sin\theta\cos\theta = \sin 2\theta$$

$$(x')^2 + (y')^2 = 1.$$

$$A = \int_0^{\pi} 2\pi y \sqrt{(x')^2+(y')^2}\, d\theta = 2\pi \int_0^{\pi} \sin^2\theta\, d\theta$$

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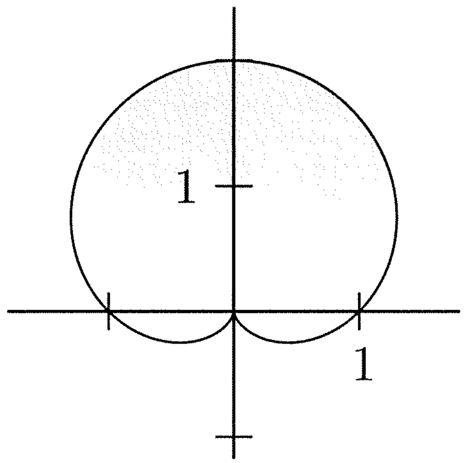
25.(6 pts.) Find the area inside the cardioid $r = 1 + \sin\theta$.



(a) $\frac{3}{2}$ (b) 2π (c) 2 (d) $3\pi + \ln 4$ (e) $\frac{3\pi}{2}$

$$A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (1+\sin\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(1 + 2\sin\theta + \sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(1 + 2\sin\theta + \frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} + 2\cos\theta - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left(\frac{3}{2}\theta - 2\sin\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2}\cdot\frac{3}{2}\cdot 2\pi = \frac{3\pi}{2}$$